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Continuity conditions at bending and shearing interfaces of rigid, perfectly plastic structural elements

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Abstract

Bending and shearing hinges idealised as discontinuity interfaces in rigid, perfectly plastic analysis are two important ways to consume plastic deformation energy in dynamic plastic response of a two-dimensional structural element. The continuity conditions at an interface in a rigid, perfectly plastic beam are examined in the present paper when retaining the transverse shear force in the yield condition. Various continuity conditions are suggested at stationary and moving bending and shearing interfaces which may occur during the early response stage of a rigid, perfectly plastic beam. Both regular and singular yield surfaces give the same continuity conditions. The effects of rotatory inertia on the continuity conditions are discussed. It is shown that transverse shear deformations in a rigid, perfectly plastic beam are always localized in a stationary shear hinge, which may lead to a distinct transverse shear, or Mode III, failure. All conclusions for the beam are extended to axisymmetrically loaded circular plates and cylindrical shells.

1. Introduction

It has been shown that deformation localizations in a rigid, perfectly plastic structural element when subjected to a transverse dynamic load are represented by plastic hinges during the early response phase before a membrane state is reached (Jones, 1989, 1997). These plastic hinges, including bending and shearing hinges, are basically discontinuity interfaces of the generalized displacements. For example, a bending hinge corresponds to a discontinuity interface of rotation angle, and a shear hinge is associated with the discontinuity interface of transverse displacement. The behaviours of these discontinuity interfaces are required in a dynamic response analysis using rigid-plastic method, which has been used

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widely in many structural impact applications (Jones, 1989, 1997). Furthermore, the existences of bending and shear discontinuity interfaces (or plastic hinges) imply infinite bending and shearing deformations within these hinges, which may lead to deformation localization and failure initiation in a transversely loaded structural element.

Bending and shearing hinges represent general characteristics of the dynamic plastic response of several two-dimensional structural elements under transverse load before the membrane state is reached. The continuity conditions at an interface must satisfy the conservation of momentum across the interface and the kinematic admissibility of the motion at this interface. A large amount of literature now exists for the dynamic plastic bending response of structural elements (Lee and Symonds, 1952; Hopkins and Prager, 1954; Hodge, 1955), in which the continuity conditions at bending hinges in structural elements were discussed. These works are also summarized by Jones (1997).

It has been shown that a transverse shear force plays an important role in the dynamic plastic response of structures (Symonds, 1968; Jones, 1997) when subjected to an intensive dynamic loading. Symonds (1968) discussed the continuity conditions at an interface in a beam when the influence of the transverse shear force is retained in the yield condition, and these results have been used widely in the analyses of dynamic plastic response of beams (Jones and de Oliveira, 1979; Nonaka, 1977; Li and Jones, 1995a). However, Symonds' conclusions are based on a particular square yield surface and need to be re-examined when the stress components at a rigid-plastic interface lie at a singular position of a yield surface. The dynamic and kinematic continuity conditions at a discontinuity interface in axisymmetrically loaded circular plates and cylindrical shells have been obtained by Jones and de Oliveira (1980, 1983) using generalized stresses and strains when transverse shear force and bending moments are retained in the yield surface, which were formulated from the general dynamic and kinematic continuity conditions across a discontinuity surface in continuum (Nowacki, 1978). Some further investigations on continuity conditions across a discontinuity surface in an elastoplastic solid have been discussed by Drugan and Shen (1987) and Nemat-Nasser and Gao (1988).

Although, the above mentioned continuity conditions at an interface have been used widely in theoretical and numerical analyses into the dynamic plastic response of structural elements, and predict reasonable results, they were, nevertheless, stated neatly for the convenient use in each case of rigid-plastic interfaces existing in rigid, perfectly plastic structural elements described with generalized stresses and strains. Symonds (1968) discussed the properties of both moving and stationary bending and shearing interfaces in rigid, perfectly plastic beams. This work might be extended to axisymmetrically loaded, rigid, perfectly plastic circular plates and cylindrical shells by a parallel analysis and to a general yield condition.

The purpose of this paper is to extend Symonds' results for a square yield curve between the bending moment and the transverse shear force in a beam to a more general yield condition which may be a regular or a singular one, and to present general continuity conditions for both shear and bending rigid-plastic interfaces. Furthermore, the conclusions for a rigid, perfectly plastic beam are proved to be valid for axisymmetrically loaded circular plates and cylindrical shells which are made from rigid, perfectly plastic materials, even though different generalized stresses are involved in the yield condition. One significant result of the present work is to prove that plastic shear deformation in a two-dimensional metal element has the localization feature when its plastic hardening modulus is relatively smaller than its elastic modulus, i.e., when rigid, perfectly plastic simplification is applicable. Mode III failure observed and defined originally by Menkes and Opat (1973) is the result of localized plastic shear deformation.

2. Basic assumptions and equations

When the local theory (de Oliveira and Jones, 1978) is used in an analysis for the dynamic plastic response of beams without axial deformations, the generalized stresses and their conjugated generalized strains are M, Q, κ and γ, respectively, which should satisfy the material stability postulate proposed by Drucker (1951, 1964). Therefore, for a regular yield surface $\boldsymbol{f}(M,Q) = 0$, we have

$$\mathrm{d}\kappa = \mathrm{d}\lambda \frac{\partial \boldsymbol{f}}{\partial M} \tag{1a}$$

and

$$\mathrm{d}\gamma = \mathrm{d}\lambda \frac{\partial \boldsymbol{f}}{\partial Q} \tag{1b}$$

in which $\mathrm{d}\lambda = 0$ for $\boldsymbol{f} < 0$ (before yielding, or rigid case) or $\boldsymbol{f} = 0$ and $\mathrm{d}\boldsymbol{f} < 0$ (unloading case), and $\mathrm{d}\lambda \geqslant 0$ for $\boldsymbol{f} = 0$ and $\mathrm{d}\boldsymbol{f} = 0$ (neutral loading case[1]) when the beam is made from rigid, perfectly plastic material. In the plastic loading case, the flow directions of the generalized strains are in a direction normal to a regular yield surface, which has a unique normal direction at every point.

For a singular yield surface consisting of a number of n regular yield functions $\boldsymbol{f}_p(M, Q)$, $p = 1, \ldots, n$, plastic yielding occurs as soon as at least one of the functions $\boldsymbol{f}_p$ is zero. It is evident that all the points on the yield surface, when only one function $\boldsymbol{f}_p = 0$, are regular and the corresponding generalized plastic strain increments are the same as those for a completely regular yield surface $\boldsymbol{f}_p = 0$. In those situations when the generalized stresses are at the intersection of two or more surfaces $\boldsymbol{f}_p = 0$, the flow rule of plastic yielding for such a singular point is obtained by combining each yield function separately according to Koiter's suggestion (Koiter, 1953), which leads to

$$\mathrm{d}\kappa = \sum_1^n \mathrm{d}\lambda_p \frac{\partial \boldsymbol{f}_p}{\partial M} \tag{2a}$$

and

$$\mathrm{d}\gamma = \sum_1^n \mathrm{d}\lambda_p \frac{\partial \boldsymbol{f}_p}{\partial Q} \tag{2b}$$

where,

$$\mathrm{d}\lambda_\mathrm{p} = 0, \quad \text{for}\, \boldsymbol{f}_\mathrm{p} < 0, \quad \text{(rigid case)} \quad \text{or}\, \boldsymbol{f}_\mathrm{p} = 0 \quad \text{and}\ \mathrm{d}\boldsymbol{f}_\mathrm{p} < 0 \quad \text{(unloading case)},$$

$$\mathrm{d}\lambda_\mathrm{p} \geqslant 0 \quad \text{for}\, \boldsymbol{f}_\mathrm{p} = 0 \quad \text{and}\ \mathrm{d}\boldsymbol{f}_\mathrm{p} = 0 \quad \text{(neutral loading case).} \tag{3}$$

Generally speaking, the flow directions of the generalized strains are uncertain at a singular point, but are bounded by the normal directions of each adjacent regular surface. This uncertainty is avoided by using the dynamic equations and the kinematic admissibility of the deformation field.

In the following analyses, it is assumed that the plane cross-section of a beam remains plane before and after loading. Furthermore, we assume that the final deformation of a beam is not influenced by the loading sequence of the bending moment and the transverse shear force, and therefore, the deflection of

[1] For a perfectly plastic material, the initial yield surface cannot expand, therefore, only the neutral loading case exists.

a beam consists of two parts which are related to the bending and shearing deformations

$$w = w_{\mathrm{b}} + w_{\mathrm{s}}. \tag{4}$$

Differentiation of eqn (4) with respect to time leads to

$$\frac{\partial w}{\partial t} = \dot{w} = \dot{w}_{\mathrm{b}} + \dot{w}_{\mathrm{s}}. \tag{5}$$

From eqns (4) and (5), we define the following quantities for small deformations

$$\frac{\partial w}{\partial t} = \Psi + \gamma \tag{6a}$$

$$\frac{\partial^2 w}{\partial t\, \partial x} = \omega + \frac{\partial \gamma}{\partial t} \tag{6b}$$

and

$$\kappa = \frac{\partial \Psi}{\partial x}, \tag{6c}$$

where, $\Psi = \partial w_b/\partial x$, $\omega = \partial \Psi/\partial t$ and κ are rotation angle, angular velocity and the curvature associated with bending, respectively, and $\gamma = \partial w_s/\partial x$ is the transverse shear strain. It has been shown that the curvature change and transverse shear defined by eqns (6a) and (6c) are consistent with equilibrium equations (Jones, 1997). It is generally accepted that these assumptions are valid for beams in small deformation. However, experimental evidences are expected to clarify the actual limitation of these assumptions.

The conservation of momentum across a discontinuity interface requires (Jones, 1997)

$$[Q]_{\xi} = -m\dot{\xi}\left[\frac{\partial w}{\partial t}\right]_{\xi} \tag{7a}$$

and

$$[M]_{\xi} = -I_r\dot{\xi}\left[\frac{\partial \Psi}{\partial t}\right]_{\xi}. \tag{7b}$$

in which, ξ is the position of an interface, and $\dot{\xi}$ is the propagation velocity of a moving interface, m is the mass per unit length of a beam, I_{r} is the rotatory inertia per unit length of a beam, and $[\,]_{\xi}$ is the difference of a physical quantity across the discontinuity interface.

3. Results for a regular yield surface

Now, the kinematic continuity conditions across a moving interface are (Jones, 1997; Symonds, 1968)

$$[w]_{\xi} = 0 \tag{8a}$$

and

$$[\Psi]_{\xi} = 0, \tag{8b}$$

which imply that

$$\left[\frac{\partial w}{\partial t}\right]_\xi + \dot{\xi}\left[\frac{\partial w}{\partial x}\right]_\xi = 0 \tag{9a}$$

and

$$\left[\frac{\partial \Psi}{\partial t}\right]_\xi + \dot{\xi}\left[\frac{\partial \Psi}{\partial x}\right]_\xi = 0 \tag{9b}$$

which allows eqns (7a) and (7b) to be written as

$$[Q]_\xi = m\dot{\xi}^2[\gamma]_\xi \tag{10a}$$

and

$$[M]_\xi = I_\mathrm{r}\dot{\xi}^2[\kappa]_\xi \tag{10b}$$

when using $[\partial w/\partial x]_\xi = [\Psi]_\xi + [\gamma]_\xi = [\gamma]_\xi$.

Consider an interface moving from a perfectly plastic zone to a rigid segment, as shown in Fig. 1(a). The points on the rigid side of the interface will yield when reached by the moving interface. Eqns (10a) and (10b) may be re-written as

$$\mathrm{d}Q = m\dot{\xi}^2\mathrm{d}\gamma \tag{11a}$$

and

$$\mathrm{d}M = I_\mathrm{r}\dot{\xi}^2\mathrm{d}\kappa \tag{11b}$$

where, dA means the increment of a physical quantity A on the rigid side of an interface shown in Fig. 1(a).

For a rigid, perfectly plastic material,

$$\mathrm{d}f = \frac{\partial f}{\partial Q}\mathrm{d}Q + \frac{\partial f}{\partial M}\mathrm{d}M = 0 \tag{12}$$

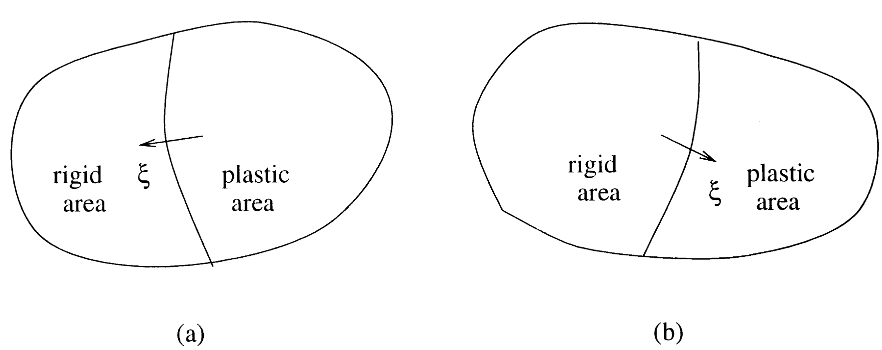

Fig. 1. Discontinuity interfaces in a rigid, perfectly plastic material: (a) moving from a perfectly plastic zone to rigid zone; (b) moving from a rigid zone to a perfectly plastic zone.

during a loading process. According to eqns (11a) and (11b) and eqns (1a) and (1b), (12) gives

$$\dot{\xi}^2 \,\mathrm{d}\lambda\left[m\left(\frac{\partial f}{\partial Q}\right)^2+I_r\left(\frac{\partial f}{\partial M}\right)^2\right]=0 \tag{13}$$

which indicates that $\mathrm{d}\lambda=0$ because $\dot{\xi}\neq 0$, $m>0$, $\mathrm{I_r}>0$ and $\partial f/\partial Q$ and $\partial f/\partial M$ cannot equal zero simultaneously. Thus, $\mathrm{d}\gamma=\mathrm{d}\kappa=0$ or $[\gamma]_\xi=[\kappa]_\xi=0$, which means that there is no discontinuity at such an interface in a rigid, perfectly plastic beam. It transpires that a discontinuity in γ or κ cannot develop at an interface which moves from a plastic zone to a rigid segment. This conclusion has been presented by Symonds (1968) for a rigid, perfectly plastic beam when the generalised stresses are on the regular points of a square yield surface.

If an interface in a rigid, perfectly plastic beam moves from the rigid segment to a plastic zone, as shown in Fig. 1(b), the plastic side of the interface will become rigid as the interface moves across it. This is an unloading process for the region on the plastic side of interface, which, therefore, requires $\mathrm{d}\lambda=0$ in eqns (1a) and (1b), or

$$[\gamma]_\xi=[\kappa]_\xi=0 \tag{14}$$

and

$$[Q]_\xi=[M]_\xi=0. \tag{15}$$

In this case, $\dot{\xi}$ may not be zero, as observed in many theoretical results for the dynamic plastic response of beams. One example is the second response phase for the simply supported beam in Section 3.2 of Jones and de Oliveira (1979).

If $I_r=0$, eqn (13) with $\dot{\xi}\neq 0$ requires $\partial f/\partial Q=0$ when $\mathrm{d}\lambda\neq 0$, which leads to $[\gamma]_\xi=0$. And, therefore, the shear interface is stationary. However, in this case, $[\kappa]_\xi$ may not equal zero as an interface in a rigid, perfectly plastic beam moves from a plastic zone to a rigid segment. The appendix of Zhu, et al. (1986) gave an example of this case for the bending response of a simply supported beam subjected to a general pulse pressure loading. Similar results for a circular plate was obtained by Youngdahl (1971).

Eqns (6a), (8b) and (14) give $[\partial w/\partial x]_\xi=[\Psi]_\xi+[\gamma]_\xi=0$, and, therefore,

$$\left[\frac{\partial w}{\partial t}\right]_\xi=0$$

according to eqn (9a) whether I_r equals zero or not. Thus, the kinematic continuity conditions across a moving interface in a rigid, perfectly plastic beam (both cases in Fig. 1) are

$$[w]_\xi=0 \tag{16a}$$

$$[\dot{w}]_\xi=0, \tag{16b}$$

which, together with eqn (15) have been used in the previous theoretical analyses reported by Symonds (1968), Jones and de Oliveira (1979), Nonaka (1977), and Li and Jones (1995a). It may be shown that eqns (16a) and (16b) are the sufficient and necessary conditions for the satisfaction of eqns (8a) and (8b) in the case of Fig. 1(a), and eqns (16a) and (16b) are equivalent to eqns (8a) and (8b) for the case of Fig. 1(b) under the assumption that $[\gamma]_\xi=0$ when $\dot{\xi}\neq 0$ i.e., a shear interface is always stationary, which is an important conclusion of previous analyses. Thus, eqns (16a) and (16b) may be used, instead of eqns (8a) and (8b), as kinematic continuity conditions across the interface moving from a plastic zone to a rigid segment in a rigid, perfectly plastic beam.

The above analyses show that a shear interface in a rigid, perfectly plastic beam is alwasys stationary whether or not $I_r = 0$. This implies that the transverse shear deformation is always localized in its initially formed zone throughout the dynamic plastic response of beams. The characteristic length of this zone will be very small compared with the beam length, because the size of the shear deformation zones does not increase during the subsequent beam response. All transeverse shear deformations will be localized in this zone, which, therefore, may be idealized as a plane with transverse shear sliding. The sliding displacement at an idealized plane is the relative displacement at two sides of the transverse shear deformation zone, as illustrated in Fig. 2(a, b). Due to the antisymmetric property of the transverse shear force about the midplane of a shear deformation zone, the kinematic continuity condition across the transverse shear sliding plane is

$$[\Psi]_\xi = 0, \tag{17}$$

and eqn (15) should be satisfied for either $I_r \neq 0$ or $I_r = 0$.

It should be noted that the assumption of shear sliding does not mean that the actual severance occurs at the shear sliding interface. $[w]_\xi \neq 0$ is an idealized result when we neglect the size of the transverse shear deformation zone, as discussed above.

For a stationary bending interface, only the transverse deflection needs to be continuous, i.e., $[w]_\xi = 0$ which is equivalent to $[\dot{w}]_\xi = 0$ according to eqn (9a). Eqns (7a) and (7b) require $[Q]_\xi = [M]_\xi = 0$.

In summary, the continuity conditions at an interface in a rigid, perfectly plastic beam may be expressed as

1. Moving bending interface

$$[Q]_\xi = 0, \quad [M]_\xi = 0, \quad [\dot{w}]_\xi = 0, \quad [w]_\xi = 0; \tag{18a)–(18d}$$

2. Stationary bending interface

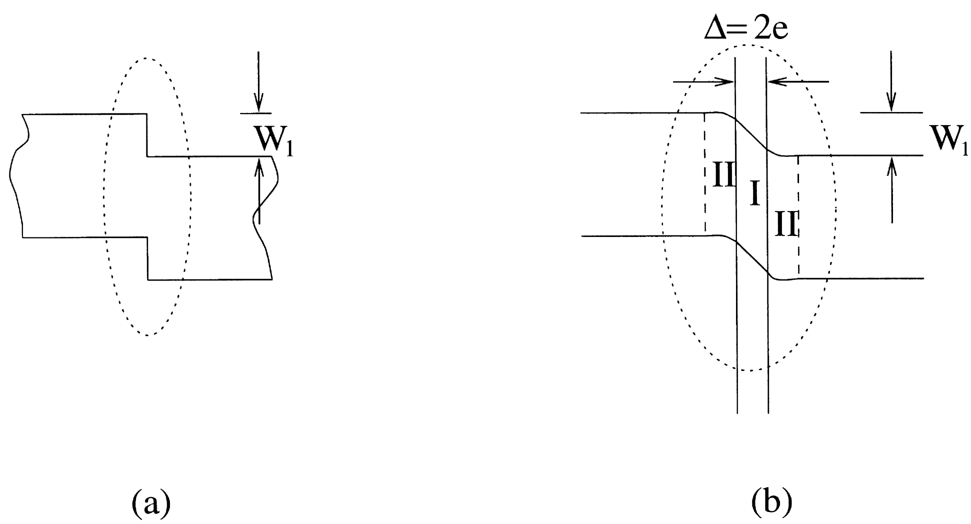

Fig. 2. (a) Idealized shear hinge; (b) construction of a shear hinge.

$$[Q]_\xi = 0, \quad [M]_\xi = 0 \text{ and } [\dot{w}]_\xi = 0 \quad \left(\text{or } [w]_\xi = 0\right); \tag{19a)–(19c}$$

3. Stationary shear slides

$$[Q]_\xi = 0, \quad [M]_\xi = 0 \text{ and } [\Psi]_\xi = 0 \left(\text{or } \left[\frac{\partial w}{\partial x}\right]_\xi = 0\right); \tag{20a)–(20c}$$

4. Stationary bending and shear interface

$$[Q]_\xi = 0 \tag{21a}$$

and

$$[M]_\xi = 0 \tag{21b}$$

All these continuity conditions in each case are identical to those presented by Li and Jones (1995a) and lead to the special cases examined by previous studies on the transverse bending and shear responses of a rigid, perfectly plastic beam.

In the dynamic plastic response of a structure element subjected to lateral loads, bending deformation is always the important mode to absorb kinematic energy. However, with the increase of loading rate and intensity, transverse shear becomes more important and appears at an early stage. Membrane state might be reached for large loads at a final stage. Menkes and Opat's (1973) experimental results gave good examples. The absorbed energies by different response modes for plates are shown by Corran et al. (1983) and Jones et al. (1997).

During the early response stage, rotatory inertia associated with bending movement may contribute to the global response of a beam, which has been studied by several authors (Jones, 1989, 1997). It was concluded that rotatory inertia effects are not of practical significance, while transverse shear forces are more important for the dynamic case than they are for static loads (Symonds, 1968; Jones, 1989). The current study indicates that rotatory inertia does not influence the shear hinge feature, but a kinky bending hinge might be formed when neglecting the rotatory inertia.

4. Results for a singular yield surface

If one of the parameters $d\lambda_p = 0$ ($p = 1, 2$) for a singular yield surface, then the associated terms in eqns (2a) and (2b) are eliminated, and the remaining terms in eqns (2a) and (2b) are the same as eqns (1a) and (1b). Therefore, the results for this situation is the same as those for a regular yield surface.

In the case when $d\lambda_p > 0$ ($p = 1, 2$), then only the loading situation in Fig. 1(a) requires discussion. The results for the unloading situation in Fig. 1(b) are similar to those for a regular yield surface.

Now, eqn (3) with $df_p = 0$ for neutral loading requires

$$\frac{\partial f_1}{\partial Q} dQ + \frac{\partial f_1}{\partial M} dM = 0, \tag{22a}$$

and

$$\frac{\partial f_2}{\partial Q} dQ + \frac{\partial f_1}{\partial M} dM = 0, \tag{22b}$$

Eqns (22a) and (22b) with eqns (11a), (11b), (2a) and (2b) may be expressed in the form

$$\left[m\left(\frac{\partial f_1}{\partial Q}\right)^2+I_r\left(\frac{\partial f_1}{\partial M}\right)^2\right]d\lambda_1+\left[m\frac{\partial f_1}{\partial Q}\frac{\partial f_2}{\partial Q}+I_r\frac{\partial f_1}{\partial M}\frac{\partial f_2}{\partial M}\right]d\lambda_2=0, \tag{23a}$$

and

$$\left[m\frac{\partial f_1}{\partial Q}\frac{\partial f_2}{\partial Q}+I_r\frac{\partial f_1}{\partial M}\frac{\partial f_2}{\partial M}\right]d\lambda_1+\left[m\left(\frac{\partial f_2}{\partial Q}\right)^2+I_r\left(\frac{\partial f_2}{\partial M}\right)^2\right]d\lambda_2=0, \tag{23b}$$

provided $\dot{\xi} \neq 0$.

The condition for $d\lambda_p \neq 0$ ($p = 1, 2$) is

$$\left[m\left(\frac{\partial f_1}{\partial Q}\right)^2+I_r\left(\frac{\partial f_1}{\partial M}\right)^2\right]\left[m\left(\frac{\partial f_2}{\partial Q}\right)^2+I_r\left(\frac{\partial f_2}{\partial M}\right)^2\right]-\left(m\frac{\partial f_1}{\partial Q}\frac{\partial f_2}{\partial Q}+I_r\frac{\partial f_1}{\partial M}\frac{\partial f_2}{\partial M}\right)^2=0$$

or

$$\frac{\partial f_1}{\partial Q}\frac{\partial f_2}{\partial M}-\frac{\partial f_1}{\partial M}\frac{\partial f_2}{\partial Q}=0 \tag{24}$$

when $m \neq 0$ and $I_r \neq 0$.

Eqn (24) implies that

$$\vec{n}_1\times\vec{n}_2=\left(\frac{\partial f_1}{\partial Q}\vec{i}+\frac{\partial f_1}{\partial M}\vec{j}\right)\times\left(\frac{\partial f_2}{\partial Q}\vec{i}+\frac{\partial f_2}{\partial M}\vec{j}\right)=0 \tag{25}$$

where, $\vec{n}_1$ and $\vec{n}_2$ are the normal directions of $f_1(M, Q) = 0$ and $f_2(M, Q) = 0$ at the intersection point, respectively, and $\vec{i}$ and $\vec{j}$ are unit vectors of the local orthogonal coordinates. eqn (25) implies that $\vec{n}_1$ is parallel to $\vec{n}_2$. Therefore, the intersection point of $f_1(M, Q) = 0$ and $f_2(M, Q) = 0$ must be a regular point if eqn (24) is satisfied. This situation has been discussed in Section 3. Thus,

$$\frac{\partial f_1}{\partial Q}\frac{\partial f_2}{\partial M}-\frac{\partial f_1}{\partial M}\frac{\partial f_2}{\partial Q}\neq 0 \tag{26}$$

which leads to a unique zero solution for eqns (23a) and (23b)

$$d\lambda_1 = 0 \quad \text{and } d\lambda_2 = 0$$

and thus, the following conditions are reached

$$[\gamma]_\xi = [\kappa]_\xi = 0 \quad \text{and} \quad [Q]_\xi = [M]_\xi = 0 \tag{27a)–(27d}$$

at an interface in a rigid, perfectly plastic beam by using eqns (2a) and (2b) and eqns (10a) and (10b).

It transpires that all the conclusions for a moving interface at a singular yield point for a rigid, perfectly plastic beam are the same as those for the regular yield surface studied in Section 3 when $I_r \neq 0$. This conclusion is also true for the continuity conditions at both moving and stationary interfaces.

If $I_r = 0$, then eqn (10b) gives

$$[M]_\xi = 0 \quad (\text{or } dM = 0), \tag{28}$$

while, eqns (22a) and (22b) become

$$\frac{\partial f_1}{\partial Q}\mathrm{d}Q = 0, \tag{29a}$$

and

$$\frac{\partial f_2}{\partial Q}\mathrm{d}Q = 0, \tag{29b}$$

If $\partial f_1/\partial Q = 0$ and $\partial f_2/\partial Q = 0$ occur simultaneously, $[\gamma]_\xi = 0$ according to eqn (2b). Otherwise, i.e., $\partial f_1/\partial Q$ and $\partial f_2/\partial Q$ cannot equal zero simultaneously, eqns (29a) and (29b) will lead to $\mathrm{d}Q = 0$, or

$$[Q]_\xi = 0 \tag{30}$$

which gives

$$[\gamma]_\xi = 0 \tag{31}$$

from eqn (10a) if $\dot{\xi} \neq 0$. Therefore, $\dot{\xi} = 0$ for $[\gamma]_\xi \neq 0$.

Again, a shear interface must be stationary, and eqn (10b) implies the possibility that a bending interface may move, which is the same conclusion as that reached for a regular yield surface. Thus, eqns (18)–(21) are valid for both regular and singular yield surfaces.

5. Continuity conditions for circular plates and cylindrical shells

The yield surface, which is shown in Fig. 3, has the following properties

$$\frac{\partial f_1}{\partial P} = 1, \quad \frac{\partial f_1}{\partial M} = 0, \quad \frac{\partial f_1}{\partial Q} = 0, \quad \frac{\partial f_2}{\partial P} = 0 \tag{32a)–(32d}$$

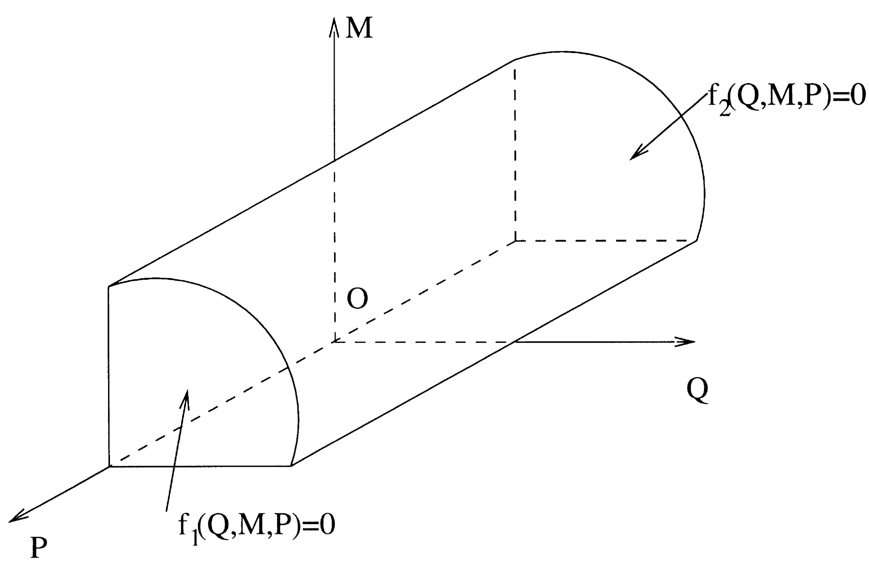

Fig. 3. Yield surface.

in which, $P = M_\theta$ (or N) for circular plate (or cylindrical shell); where M_θ is the circumferential bending moment in a circular plate and N is the circumferential member force in a cylindrical shell.

For both an axisymmetrically loaded circular plate and an axisymmetrically loaded cylindrical shell, eqns (1)–(11) will be satisfied according to the basic equations in Jones and de Oliveira (1980, 1983).[2] Furthermore,

$$\kappa_\theta = \frac{\Psi}{\mathrm{r}}, \quad \text{for a circular plate} \tag{33a}$$

and

$$\varepsilon_\theta = -\frac{w}{R}, \quad \text{for a cylindrical shell} \tag{33b}$$

where, r is the radial coordinate of circular plate and R is the mean radius of a cylindrical shell.

In the following discussion, $f_2(Q, M, P) = 0$ is assumed to be a regular surface. However, it does not influence the results when $f_2(Q, M, P) = 0$ is a singular one because of the conclusions obtained in Section 4.

The assumption of a rigid, perfectly plastic material requires

$$\mathrm{d}f_1 = \frac{\partial f_1}{\partial Q}\mathrm{d}Q + \frac{\partial f_1}{\partial M}\mathrm{d}M + \frac{\partial f_1}{\partial P}\mathrm{d}P = \frac{\partial f_1}{\partial P}\mathrm{d}P = \mathrm{d}P = 0, \tag{34a}$$

and

$$\mathrm{d}f_2 = \frac{\partial f_2}{\partial Q}\mathrm{d}Q + \frac{\partial f_2}{\partial M}\mathrm{d}M + \frac{\partial f_2}{\partial P}\mathrm{d}P = \frac{\partial f_2}{\partial M}\mathrm{d}M + \frac{\partial f_2}{\partial Q}\mathrm{d}Q = 0, \tag{34b}$$

while, the plastic flow rule from eqn (2) gives

$$\mathrm{d}\gamma = \mathrm{d}\lambda_1\frac{\partial f_1}{\partial Q} + \mathrm{d}\lambda_2\frac{\partial f_2}{\partial Q} = \mathrm{d}\lambda_2\frac{\partial f_2}{\partial Q} \tag{35a}$$

$$\mathrm{d}\kappa = \mathrm{d}\lambda_1\frac{\partial f_1}{\partial M} + \mathrm{d}\lambda_2\frac{\partial f_2}{\partial M} = \mathrm{d}\lambda_2\frac{\partial f_2}{\partial M} \tag{35b}$$

and

$$\mathrm{d}g = \mathrm{d}\lambda_1\frac{\partial f_1}{\partial P} + \mathrm{d}\lambda_2\frac{\partial f_2}{\partial P} = \mathrm{d}\lambda_1 \tag{35c}$$

where, $g = \kappa_\theta$ (or ε_θ).

It is evident that eqns (34b), (35a) and (35b) together with eqns (1)–(10) are the same as the corresponding equations for a beam. The extra eqns (33a) and (33b) may be satisfied by adjusting $\mathrm{d}\lambda_1$ which has no influence on $\mathrm{d}\kappa$ and $\mathrm{d}\gamma$. Thus, the continuity conditions for both a circular plate and a cylindrical shell are the same as those for a beam, except for the additional conditions from eqn (34a)

$$[P]_\xi = 0. \tag{36}$$

[2] For example, eqns (7a) and (7b) and eqns (9a) and (9b) here are the same as eqns (4b) and (4a) and eqns (6b) and (6a) in Jones and de Oliveiria (1980, 1983), respectively, when M, κ and x are replaced by M_r, k_r and r for circular plate.

It should be noted that the dynamic plastic response for circular plates and cylindrical shells usually lead to $f_1(Q, M, P) = 0$ and $\mathrm{d}g = \mathrm{d}\lambda_1 > 0$ throughout the entire area of a plate or a shell. In this case, the continuity conditions (eqns (18)–(21)) are not related to an interface between rigid regions and plastic zones in a rigid, perfectly plastic material, but to an interface within a plastic deformation zone. These conclusions have been examined for the dynamic plastic analyses of circular plates (Li and Jones, 1994) and cylindrical shells (Li and Jones, 1995b).

6. Discussion

Rigid, perfectly plastic idealization has been used successfully to predict the dynamic plastic responses of various structural elements. An important concept in this analytical method is the plastic hinge, through which external or kinetic energy is consumed by plastic dissipation. Material failures are observed frequently at localized deformation zones, which are formed by bending and shear hinges or their combinations, and sometimes, associated with the membrane deformation. The conclusion of the present paper shows that a shear hinge behaves like a 'deformation trap' because of its stationary feature. Many evidences have shown that kinetic energy is easier to be consumed in a plastic shear hinge, if it can be initiated, than in a bending hinge or a membrane state (Jones, 1997). Thus, localized shear response and possible shear failure become the dominant mode when the transverse shear conditions are satisfied, which are normally associated with dynamic loads with sufficient intensities.

A rigid, perfectly plastic analysis is an efficient way to give an approximate estimation for inelastic structural responses. The shortcoming of a rigid, perfectly plastic analysis is its limitation to give deformation distributions within a plastic hinge although some simplified methods have been introduced to overcome this difficulty. A plastic bending response appears normally during the early response phase when the membrane state has not been developed. Nonaka (1967) used slip line theory to examine the plastic hinges in fully clamped beams when the influence of the transverse shear stress on plastic yielding is neglected. It was found that the mean length of a plastic bending hinge depends on the membrane influence. Shen and Jones (1992) considered the influence of transverse shear and an approximate linear relationship between dimensionless hinge length and dimensionless dissipation density of plastic shear work within a hinge was suggested based on Menkes and Opat's (1973) experimental results. Most of the existing works, unfortunately, concern the construction of a bending hinge, except Wang and Jones (1996) who proposed an one-dimensional transverse shear propagation analysis based on rigid, plastic strain hardening model. Recently, the formation of a shear localization in two-dimensional elements was studied using FE simulation (Li and Jones, 1998a), in which shear hinge lengths for different structural elements and their valid application range were clarified. In fact, several studies have used these results to obtain shear strains within the shear hinge in order to predict failure initiation (Wen et al., 1995; Wen and Jones, 1996; Jones et al., 1997; Li and Jones, 1998b).

Plastic shear deformations are always found around 'hard point' including supports and loading periphery (Duffy, 1989), and localised into a narrow zone idealized as a plastic shear hinge. Both experimental (Menkes and Opat, 1973; Ross et al., 1977) and analytical (Symonds, 1968; Jones, 1997) works suggested that a plastic shear hinge can be initiated by rapid and high intensity loading during the early response phase. A good example of a typical shear dominant zone was given by Zener (1948), which indicates that a uniform distribution of simple shear is the idealised geometry feature of a plastic shear hinge. The existences of bending and shear hinges in a structural element subjected to lateral impact load have been shown by several experimental evidences. For example, propagation of plastic bending hinge in an impulsively loaded beam was presented by Florence and Firth (1965), where stationary plastic bending hinges were developed at supports and two travelling bending hinges moved toward the mid-point of the beam. Menkes and Opat (1973) noticed three different response and failure modes with increasing impulsive intensities, i.e.,

i.	Mode I	large inelastic deformation,
ii.	Mode II	tearing (tensile failure),
iii.	Mode III	transverse shear failure,

which have been studied by Jones (1976, 1989) using rigid, perfectly plastic analyses where the continuity conditions obtained in the present paper were employed. Shear hinges were found and measured by Shadbolt et al. (1983) at the projectile impact periphery in studying plate perforations, which has been used to predict failure initiation in plate perforation (Liu and Stronge, 1995).

Plastic bending and shear hinges are based on the rigidity assumption in rigid, perfectly plastic analysis. However, more and more evidences have shown that elasticity may play an important role in structural response (Symonds and Fleming, 1984; Symonds and Yu, 1985; Reid and Gui, 1987; Yu, 1993; Yu et al., 1997). The formation of a plastic hinge results from the interaction of reflected elastoplastic waves. Thus, the validity of current conclusions are restricted within the valid limitaion of rigid, perfectly plastic assumption.

Loading discontinuities, which may appear due to the existence of concentrated loads, are not considered in basic eqns (7a) and (7b). The conclusions obtained in the present paper are only applicable to pressure loads with finite intensity. However, concentrated load is the idealization of a distributed pressure load with high intensity and small acting area. When the loading area dimension is larger than the characteristic size of a plastic hinge and the loading intensity is finite, there is no load discontinuties across a hinge length. But, with the increase of load intensity, local phenomena like indentation may occur, which are not considered in the analysis. Thus, special attention should be paid in these cases (Jones et al., 1997).

Although plastic bending and shear hinges have been used widely in dynamic plastic response of structural elements to consume plastic energies and to propagate plastic deformations, there is a paucity of experiments to examine the existence and the feature of bending and shear hinges. Furthermore, when a rigid, perfectly plastic model is used in failure analysis, the details of a plastic hinge are necessary. Thus, further works are required to study the dynamic features, geometrical structures and forming processes of plastic bending and shear hinges in structural elements, both experimentally and numerically.

7. Conclusions

A complete set of continuity conditions at the rigid–plastic interface is proposed in the present paper for several structural elements, made form rigid, perfectly plastic material. It is shown that Symonds' conclusions (Symonds, 1968) on beam for a square yield curve are valid for a general yield curve which satisfies Koiter's assumption (Koiter, 1953). This implies that any transverse shear interfaces are always stationary and transverse shear deformations are localized in a narrow zone which is idealized as a zero-size plane with transverse shear sliding, called shear hinge. The results obtained in the present paper are valid for both beams and axisymmetrically loaded circular plates and cylindrical shells during their response phase before membrane state starts.

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References

Corran, R.S.J., Shadbolt, P.J., Ruiz, C., 1983. Impact loading of plates—an experimental investigation. Int. J. Impact Engng 1, 3–22.
de Oliveira, J.G., Jones, N., 1978. Some remarks on the influence of transverse shear on the plastic yielding of structures. Int. J. Mech. Sci. 20, 759–765.
Drucker, D.C. 1951. A more fundamental approach to plastic stress–strain relations. In: Sternberg, E. (Ed.), Proceedings of First U.S. Congress of Applied Mechanics. ASME, New York, pp. 487–491.
Drucker, D.C., 1964. On the postulate of stability of material in the mechanics of continua. J. de Mecanique 3, 235.
Drugan, W.J., Shen, Y., 1987. Restrictions on dynamically propagating surfaces of strong discontinuity in elastic–plastic solids. J. Mech. Phys. Solids 35, 771–787.
Duffy, T.A. 1989. Dynamic rupture of shells. In: Wierzbicki, T., Jones, N. (Eds.), Structural Failure. John Wiley and Sons, New York, pp. 161–192.
Florence, A.L., First, R.D., 1965. Rigid-plastic beams under uniformly distributed impulses. ASME, J. Appl. Mech. 32, 481–488.
Hodge Jr., P.G., 1955. Impact pressure loading of rigid-plastic cylindrical shells. J. Mech. Phys. of Solids 3, 176–188.
Hopkins, H.G., Prager, W., 1954. On the dynamics of plastic circular plates. ZAMP 5, 317–330.
Jones, N., 1989. Recent studies on the dynamic plastic response of structures. Appl. Mech. Rev. 42, 95–115.
Jones, N., 1997. Structural Impact. Cambridge University Press, Cambridge.
Jones, N., de Oliveira, J.G., 1979. The influence of rotatory inertia and transverse shear on the dynamic plastic behaviour of beams. ASME, J. Appl. Mech. 46, 303–310.
Jones, N., de Oliveira, J.G., 1980. Dynamic plastic response of circular plates with transverse shear and rotaory inertia. ASME, J. Appl. Mech. 47, 27–34.
Jones, N., de Oliveira, J.G., 1983. Impulsive loading of a cylindrical shell with transverse shear and rotatory inertia. Int. J. Solids Struct. 19, 263–279.
Jones, N., Kims, S.B., Li, Q.M., 1997. Response and failure of ductile circular plates struck by a mass. ASME, J. of Pressure Vessel Tech. 119, 332–342.
Koiter, W.T., 1953. Stress–strain relations, uniqueness and variational theorems for elastic-plastic materials with a singular yield surface. Quart. Appl. Math. 11, 350–354.
Lee, E.H., Symonds, P.S., 1952. Large plastic deformation of beams under transverse impact. ASME J. Appl. Mech. 19, 308–314.
Li, Q.M., Jones, N., 1994. Blast loading of fully clamped circular plates with transverse shear effects. Int. J. Solids Struct. 31, 1861–1876.
Li, Q.M., Jones, N., 1995a. Blast loading of fully clamped beams with transverse shear effects. Mechanics of Structures and Machines 23, 59–86.
Li, Q.M., Jones, N., 1995b. Blast loading of a "short" cylindrical shell with transverse shear effects. Int. J. Impact Engng 16, 331–353.
Li, Q.M., Jones, N., 1998a. Formation of a shear localization in structural elements under transverse dynamic loads. Impact Research Centre Report, No. IRC/170/98, Department of Engineering, University of Liverpool.
Li, Q.M., Jones, N., 1998b. Shear failure in an impulsively loaded fully clamped beam. Impact Research Centre Report, No. IRC/171/98, Department of Engineering, University of Liverpool.
Liu, D., Stronge, W.J., 1995. Perforation of rigid-plastic plate by blunt missile. Int. J. Impact Engng 16 (5/6), 739–758.
Menkes, S.B., Opat, H.J., 1973. Broken beams. Exp. Mech. 13, 480–486.
Nemat-Nasser, S., Gao, Y.C., 1988. Discontinuities in elastic-plastic solids. Mechanics of Material 7, 215–229.
Nonaka, T., 1967. Some interaction effects in a problem of plastic beam dynamics, Part 1–3. ASME, J. Appl. Mech. 34, 623–643.
Nonaka, T., 1977. Shear and bending response of a rigid-plastic beam to blast-type loading. Ingenieur-Archiv 46, 35–52.
Nowacki, W.K., 1978. Stress Waves in Non-elastic Solids. Pergamon Press.
Reid, S.R., Gui, X.G., 1987. On the elastic-plastic deformation of cantilever beams subjected to tip impact. Int. J. Impact Engng 6, 109–127.
Ross, C.A, Strickland, W.S., Sierakowski, R.L., 1977. Response and failure of simple structural elements subjected to blast loading. Shock and Vib. Digest 9, 15–26.
Shadbolt, P.J., Corran, R.S.J., Ruiz, C., 1983. A comparison of plate perforation models in the sub-ordnance impact velocity range. Int. J. Impact Engng 1, 23–49.
Shen, W.Q., Jones, N., 1992. A failure criterion for beams under impulsive loading. Int. J. Impact Engng 12, 101–121; 329.

Symonds, P.S. 1968. Plastic shear deformation in dynamic load problems. In: Heyman, J., Leckie, F.A (Eds.), Engineering Plasticity. Cambridge University Press, Cambridge, pp. 647–664.

Symonds, P.S., Fleming Jr., W.T., 1984. Parkes revised: on rigid-plastic and elastic plastic dynamic structural analysis. Int. J. Impact Engng 2, 1–36.

Symonds, P.S., Yu, T.X., 1985. Counterintuitive behavior in a problem of elastic-plastic beam dynamics. ASME, J. Appl. Mech. 52, 517–522.

Wang, L.L., Jones, N., 1996. An analysis of the shear failure of rigid-linear hardening beams under impulsive loading. Acta Mechanica Sinica (English Series) 12 (4), 338–348.

Wen, H.M., Jones, N., 1996. Low-velocity perforation of punch-impact-loaded metal plates. Trans. ASME J. Pressure Vessel Tech. 118, 181–187.

Wen, H.M., Reddy, T.Y., Reid, S.R., 1995. Deformation and failure of clamped beams under low speed impact loading. Int. J. Impact Engng 16, 435–454.

Youngdahl, C.K., 1971. Influence of pulse shape on the final plastic deformation of a circular plate. Int. J. Solids Struct. 7, 1127–1142.

Yu, T.X. 1993. Elastic effect in the dynamic plastic response of structures. In: Jones, N., Wierzbicki, T. (Eds.), Structural Crashworthiness and Failure. Elsevier, pp. 341–384.

Yu, T.X., Yang, J.L., Reid, S.R., 1997. Interaction between reflected elastic flexural waves and a plastic 'hinge' in the dynamic response of pulse loaded beams. Int. J. Impact Engng 19, 457–475.

Zener, C. 1948. The micro-mechanism of fracture. In: Sachs, G. (Ed.), Fracturing of Metals. American Society for Metals, Cleveland, Ohio, pp. 3–31.

Zhu, G., Huang, Y.G., Yu, T.X., Wang, R., 1986. Estimation of the plastic structural response under impact. Int. J. Impact Engng 4, 228–271.